A note on the super-H-theorem for a linear Boltzmann equation

1983 J. Phys. A: Math. Gen. 16 L351

(http://iopscience.iop.org/0305-4470/16/11/002)

J. Phys. A: Math. Gen. 16 (1983) L351–L353. Printed in Great Britain

LETTER TO THE EDITOR

A note on the super-H-theorem for a linear Boltzmann equation

J O Vigfusson

Institut für theoretische Physik der Universität Zürich, Schönberggasse 9, 8001 Zürich, Switzerland

Received 18 April 1983

Abstract. The super-H-theorem $(-1)^n \mathrm{d}^n H(t)/\mathrm{d}t^n \geqslant 0$, $n \geqslant 1$, $t \in [0, \infty)$, is proven to be incorrect for the Boltzmann equation in the relaxation time approximation (i.e. the homogeneous BGK model for Maxwellian molecules).

In his paper on the Bhatnagar–Gross–Krook (BGK) model, Simons (1972) numerically investigated the higher derivatives of the Boltzmann H-function for the special case of a spatially homogeneous gas of Maxwellian molecules. He showed that the first ten derivatives alternate in sign, in accord with the hypothesis of the super-H-theorem, which says

$$(-1)^n \mathrm{d}^n H(t)/\mathrm{d}t^n \geqslant 0, \qquad t \in [0, \infty), \qquad n \geqslant 1. \tag{1}$$

The super-H-theorem was first proposed in the mid-sixties (Harris 1967) and was supposed to hold quite generally for various transport and rate equations. There has been considerable support of the hypothesis coming from numerical calculations on different models (e.g. Simons 1972, Rouse and Simons 1978, Ziff *et al* 1981), but not until very recently have analytic results become available, proving the failure of the super-H-theorem for certain Boltzmann equation cases (Olaussen 1982, Lieb 1982, Garrett 1982, Vigfusson 1983a) and quite generally for master equations with a finite number of discrete states (Vigfusson 1983b, see also Vigfusson and Thellung 1982).

Here we apply the methods of Vigfusson (1983b) to show the invalidity of the super-H-theorem also in the special case of a BGK model mentioned above. This homogeneous BGK model is actually just the relaxation time approximation to the Boltzmann equation, with a constant relaxation time $1/\nu$ (see Simons 1972),

$$\partial_t f(\boldsymbol{v}, t) = -\nu(f(\boldsymbol{v}, t) - F(\boldsymbol{v})), \qquad F(\boldsymbol{v}) = c \exp[-\beta(\boldsymbol{v} - \boldsymbol{V})^2/2]. \tag{2}$$

The parameters of the Maxwell distribution $F(\boldsymbol{v})$ are chosen such that

$$\int \mathrm{d}^3v\, \boldsymbol{v}^2 F(\boldsymbol{v}) = \int \mathrm{d}^3v\, \boldsymbol{v}^2 f(\boldsymbol{v}, 0), \qquad \int \mathrm{d}^3v\, \boldsymbol{v} F(\boldsymbol{v}) = \int \mathrm{d}^3v\, \boldsymbol{v} f(\boldsymbol{v}, 0),$$
$$\int \mathrm{d}^3v\, F(\boldsymbol{v}) = 1 = \int \mathrm{d}^3v\, f(\boldsymbol{v}, 0), \tag{3}$$

so that the equation (2) has the same summation invariants as the full Boltzmann equation. The solution of (2) is

$$f(\boldsymbol{v}, t) = F(\boldsymbol{v}) - (F(\boldsymbol{v}) - f(\boldsymbol{v}, 0))\mathrm{e}^{-\nu t}. \tag{4}$$

Denote

$$g(\boldsymbol{v}) = (F(\boldsymbol{v}) - f(\boldsymbol{v}, 0))/F(\boldsymbol{v}). \tag{5}$$

Then

$$\dot{H} = \int \mathrm{d}^3 v\, \dot{f}(\boldsymbol{v}, t) \log f(\boldsymbol{v}, t)$$

$$= \int \mathrm{d}^3 v\, \nu F(\boldsymbol{v}) g(\boldsymbol{v})\, \mathrm{e}^{-\nu t} \log(1 - g(\boldsymbol{v})\mathrm{e}^{-\nu t}), \tag{6}$$

where we have used the fact that

$$\int \mathrm{d}^3 v\, (F(\boldsymbol{v}) - f(\boldsymbol{v}, 0)) \log F(\boldsymbol{v}) = 0, \tag{7}$$

as follows from (3).

In order to disprove (1), we show that $-\dot{H}$ does not satisfy the same set of conditions for $n \geqslant 0$. In order to do so, assume $f(\boldsymbol{v}, 0)$ to be only slightly different from $F(\boldsymbol{v})$, such that $|g(\boldsymbol{v})|$ is uniformly bounded. Then for t large enough the log can be expanded into uniformly convergent series:

$$\log(1 - g(\boldsymbol{v})\, \mathrm{e}^{-\nu t}) = -\sum_{k=1}^{\infty} \frac{1}{k} (g(\boldsymbol{v})\, \mathrm{e}^{-\nu t})^k \tag{8}$$

so that

$$\dot{H} = -\int \mathrm{d}^3 v\, \nu F(\boldsymbol{v}) \sum_{k=2}^{\infty} \frac{1}{k-1} (g(\boldsymbol{v})\, \mathrm{e}^{-\nu t})^k$$

or

$$-\dot{H} = \sum_{k=2}^{\infty} C_k\, \mathrm{e}^{-k\nu t}, \qquad C_k = \frac{1}{k-1} \int \mathrm{d}^3 v\, \nu F(\boldsymbol{v})(g(\boldsymbol{v}))^k. \tag{9}$$

This Dirichlet series can be written as a Laplace–Stieltjes transform of a step function $G(\lambda)$, with discontinuities of magnitude C_k at $\lambda = k\nu$, $k = 2, 3, \ldots$,

$$-\dot{H}(t) = \int_0^{\infty} \mathrm{e}^{-\lambda t}\, \mathrm{d}G(\lambda). \tag{10}$$

According to Bernstein's theorem (see Widder 1946) a function $h(t)$ satisfies the conditions $(-1)^n \mathrm{d}^n h/\mathrm{d}t^n \geqslant 0$, $n \geqslant 0$, $t \in [0, \infty)$, if and only if it is the Laplace–Stieltjes transform of a non-decreasing function $G(\lambda)$. It therefore only remains to show that the initial condition $f(\boldsymbol{v}, 0)$ can always be chosen such that $G(\lambda)$ in (10) is not non-decreasing, i.e. some of the C_k, equation (9), are negative. But this is trivial, for starting with any initial distribution $f(\boldsymbol{v}, 0)$, the sign of all C_k with k odd is changed by replacing $f(\boldsymbol{v}, 0)$ by $\bar{f}(\boldsymbol{v}, 0) = 2F(\boldsymbol{v}) - f(\boldsymbol{v}, 0)$. Furthermore $f(\boldsymbol{v}, 0)$ can always be chosen such that some of the C_k, k odd, are actually different from zero. This proves that there are always initial conditions $f(\boldsymbol{v}, 0)$ such that $H(t)$ does not satisfy the super-H-theorem.

References

Garrett A J M 1982 *J. Phys. A: Math. Gen.* **15** L351
Harris S 1967 *J. Math. Phys.* **8** 2407
Lieb E H 1982 *Phys. Rev. Lett.* **48** 1057
McElwain D L S and Pritchard H O 1969 *J. Am. Chem. Soc.* **91** 7693
Olaussen L 1982 *Phys. Rev.* A **25** 3393
Rouse S and Simons S 1978 *J. Phys. A: Math. Gen.* **11** 423
Simons S 1972 *J. Phys. A: Gen. Phys.* **5** 1537
Vigfusson J O 1983a *Physica* to appear
——1983b *Lett. Math. Phys.* to appear
Vigfusson J O and Thellung A 1982 *Phys. Lett.* **91A** 435
Widder D V 1946 *The Laplace Transform* (Princeton: University Press) ch IV
Ziff R M, Merajver S D and Stell G 1981 *Phys. Rev. Lett.* **47** 1493